Uncertainty constants and quasispline wavelets*

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Abstract

In 1996 Chui and Wang proved that the uncertainty constants of scaling and wavelet functions tend to infinity as smoothness of the wavelets grows for a broad class of wavelets such as Daubechies wavelets and spline wavelets. We construct a class of new families of wavelets (quasispline wavelets) whose uncertainty constants tend to those of the Meyer wavelet function used in construction.

Key words: wavelet function, scaling function, mask, the Meyer wavelet, a linear method of summation, de la Vallee Poussin mean, uncertainty constant

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1 Introduction

One of the main advantages of wavelet systems is the good time-frequency localization. The smoothness of wavelets is also a useful and desired property. So

$$\text{\textit{to find wavelets preserving time-frequency}} \atop \text{\textit{localization as smoothness grows}} \tag{1}$$

is a very attractive and interesting problem. In the sequel, by a wavelet we mean a function generating an orthonormal basis of $L_2(\mathbb{R})$ (see the definition in section 2). The measure of the time-frequency localization is an uncertainty constant (see the definition in section 2). So in problem (1) we are interested in uncertainty constants bounded with respect to a smoothness

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parameter. It is well known that the main classical families of wavelets contain wavelet functions with arbitrary large finite smoothness. Thus, one can investigate how a functional defined on a family of wavelets depends on the smoothness of the wavelets. Let the functional be the uncertainty constant. Unfortunately, the main classical families of wavelets lose the time-frequency localization as the smoothness of chosen wavelet function grows. More precisely, Chui and Wang [1] show that the uncertainty constants of scaling and wavelet functions tend to infinity as the smoothness of the wavelets grows for a broad class of wavelets such as for example Daubechies wavelets and spline wavelets. So Daubechies wavelets and spline wavelets don't settle (1).

Later Chui, Wang [2] and Goodman, Lee [3] construct families of nonorthogonal scaling functions and semi-orthogonal wavelet functions. These functions have optimal uncertainty constants (in the sense of Heisenberg uncertainty principle) as the smoothness parameter tends to infinity. But nothing is said about orthogonal scaling and wavelet functions in [2] and [3].

Trying to solve problem (1), Novikov [4], [5] constructs a family of modified Daubechies wavelets. The wavelet functions are compactly supported. The squared module of the modified Daubechies mask is the Bernstein polynomial. It interpolates a piecewise linear function (instead of the characteristic function of an interval as it is in the case of classical Daubechies). The smoothness of the modified Daubechies wavelet grows as the order of the Bernstein polynomial increases. The time-frequency localization of the autocorrelation function which is constructed for the scaling function of this family is preserved with the growth of the smoothness. It is still an open question whether the modified Daubechies scaling and wavelet functions preserve the time-frequency localization as the smoothness growing.

In [6], the author constructs a new wavelet family solving problem (1) for scaling functions. New scaling functions decay exponentially and their Fourier transforms decay as $O(\omega^{-l})$, like spline wavelets; the uncertainty constants of the scaling functions are uniformly bounded with respect to the smoothness parameter l. The construction is based on de la Valle-Poussin means of a function closely connected with the Meyer mask.

In the present paper, we construct a wide class of such wavelets (see Theorem 1). The new wavelet function also decays exponentially at infinity and its Fourier transform decays as $O(\omega^{-l})$, like spline wavelet; that is why it is named a quasispline wavelet function (see Definition 1). The construction is based on the linear method of summation satisfying some weak, easily done conditions (see Theorem 2). The wavelet system constructed in [6] is an

example of the quasispline wavelets. It is proven that the quasispline wavelets solve problem (1) for scaling and wavelet functions. Moreover, since the uncertainty constant for the Meyer scaling and wavelet function is bounded, a special property for the quasispline wavelets is proven. This property is stronger than the boundedness. Namely, we establish the convergence of the uncertainty constants defined for the new scaling (wavelet) functions to those of the Meyer scaling (wavelet) function used in construction as the smoothness parameter l goes to infinity. The latter result also means that the uncertainty constant is a continuous functional, where the variable of the functional is a non-orthogonal mask m_l. We also estimate the rate of the convergence. It is necessary to note that the construction of quasispline wavelets can be based not only on the Meyer mask but also on any smooth orthogonal mask m such that $m(\omega) = 1$ if $|\omega| < a$ and $m(\omega) = 0$ if $b < |\omega| < \pi$ for some $\pi/3 \leq a < b < \pi$.

2 Notations and auxiliary results

Denote by $[x]$ an integer part of a real number x. Denote by $C^k[a, b]$ a space of all k times continuously differentiable functions defined on the interval $[a, b]$ with norm $\|f\|_{W^k_\infty} := \sum_{j=0}^{k} \max_{x\in[a,b]} |f^{(j)}(x)|$, write $C^0[a, b] = C[a, b]$ and $C[-\pi, \pi] = C$.

We choose `the Fourier transform` and the reconstruction formula as

$$\widehat{f}(\omega) := \int_{\mathbb{R}} f(t)e^{-it\omega}\, dt, \quad f(t) := \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(\omega)e^{it\omega}\, d\omega$$

respectively. For the Fourier series $f \sim \frac{a_0}{2} + \sum_{n\in\mathbb{N}} a_n \cos n\omega + b_n \sin n\omega$ the sequence $(\lambda_{n,k})$, $k = 1, \dots n$, $n \in \mathbb{N}$ defines `a linear method of summation`

$$u_n(f, \omega) := \frac{a_0}{2} + \sum_{k=1}^{n} \lambda_{n,k}(a_n \cos n\omega + b_n \sin n\omega) = \int_{-\pi}^{\pi} f(x)U_n(x, \omega)\, dx,$$

where $U_n(x, \omega) := 1/2 + \sum_{k=1}^{n} \lambda_{n,k} \cos k(x - \omega)$ and terms

$$a_n := \frac{1}{\pi}\int_{\pi}^{\pi} f(\omega) \cos\, n\omega\, d\omega, \quad b_n := \frac{1}{\pi}\int_{\pi}^{\pi} f(\omega) \sin\, n\omega\, d\omega$$

are the Fourier coefficients. The following property holds true

$$u_n(f', \omega) = (u_n(f, \omega))'_\omega. \tag{2}$$

A function ψ is called `a wavelet function` if the functions $2^{j/2}\psi(2^j\cdot - k)$, $j,\ k \in \mathbb{Z}$ form an orthonormal basis of $L_2(\mathbb{R})$.

Denote by $\theta(\omega)$ some odd function equal to $\frac{\pi}{4}$ for $\omega > \frac{\pi}{3}$. Assume henceforth that $\theta(\omega)$ is a non-decreasing twice continuously differentiable function. Denote by ω_0 some parameter that varies in the interval $\frac{\pi}{3} \le \omega_0 < \frac{\pi}{2}$ and put $\omega_1 := \pi - \omega_0$. `A Meyer scaling function` φ^M is defined by

$$\widehat{\varphi^M}(\omega) := \begin{cases} 1, & |\omega| \le 2\omega_0, \\ \cos\left(\frac{\pi}{4} + \theta\left(\frac{\pi}{3(\pi - 2\omega_0)}(|\omega| - \pi)\right)\right), & 2\omega_0 < |\omega| \le 2\pi - 2\omega_0, \\ 0, & |\omega| > 2\pi - 2\omega_0. \end{cases}$$

`A Meyer mask` is a 2π-periodic function defined on $[-\pi,\ \pi]$ as follows $m^M(\omega) := \widehat{\varphi^M}(2\omega)$. It is well known (see, for example [7]) that under the above restrictions on the function θ, the uncertainty constants of for the Meyer scaling and wavelet function are bounded.

`The uncertainty constant` of f is the functional $\Delta_f \Delta_{\widehat{f}}$ such that

$$\Delta_f^2 := \|f\|^{-2}_{L^2(\mathbb{R})} \int_{\mathbb{R}} (t - t_{0f})^2 |f(t)|^2\, dt, \quad \Delta_{\widehat{f}}^2 := \|\widehat{f}\|^{-2}_{L^2(\mathbb{R})} \int_{\mathbb{R}} (\omega - \omega_{0\widehat{f}})^2 |\widehat{f}(\omega)|^2\, d\omega,$$

$$t_{0f} := \|f\|^{-2}_{L^2(\mathbb{R})} \int_{\mathbb{R}} t|f(t)|^2\, dt, \qquad \omega_{0\widehat{f}} := \|\widehat{f}\|^{-2}_{L^2(\mathbb{R})} \int_{\mathbb{R}} \omega|\widehat{f}(\omega)|^2\, d\omega.$$

The terms Δ_f, $\Delta_{\widehat{f}}$, t_{0f}, and $\omega_{0\widehat{f}}$ are called `the time radius`, `the frequency radius`, `the time centre`, and `the frequency centre` of the function f respectively.

The numbers $\pm e^{i\bar{\omega}}$ are called `the pair of symmetric roots` of the mask m if $m(\bar{\omega}) = m(\bar{\omega} + \pi) = 0$. A set $B := \{b_1, \dots, b_n\}$ of distinct complex numbers is called `cyclic` if $b_{j+1} = b_j^2$, for $j = 1, \dots, n$ and $b_{n+1} = b_1$. A cyclic set B is called `the cycle of the mask` m if $m(\omega + \pi) = 0$ for all ω such that $\exp(i\omega) = b_j$ for some $j = 1, \dots, n$. `The trivial cycle` is the set $\{1\}$. A mask is called `pure` if it has neither pairs of symmetric zeros nor cycles. The following result gives a necessary and sufficient condition for integer shifts $\varphi(\cdot + k)$, $k \in \mathbb{Z}$ of a scaling function φ to be stable (i.e., to form a Riesz basis).

Proposition 1 *[8, Corollary 3.4.15] Integer shifts of a scaling function are stable (i.e., form a Riesz basis) iff corresponding mask has neither pairs of symmetric zeros nor nontrivial cycles.*

`The Hölder exponent` α_f of a function f defined on some closed interval $[a,b]$ is

$$\alpha_f := k + \sup_{\beta\in\mathbb{R}}\{\beta \in \mathbb{R} \,|\, |f^{(k)}(x_1) - f^{(k)}(x_2)| \le C_\beta |x_1 - x_2|^\beta,\ x_1, x_2 \in [a,b]\},$$

where $k := \max_{h\in\mathbb{Z}}\{h | f \in C^h[a,b]\}$. Another characteristic of the smoothness of f is

$$\theta_{\widehat{f}} := \sup_{\beta\in\mathbb{R}}\left\{\beta \in \mathbb{R} \,\middle|\, |\widehat{f}(\omega)| \le C(|\omega|+1)^{-\beta}\right\}.$$

The smoothness characteristics we introduced are known to satisfy the inequality $\theta_{\widehat{f}} - 1 \le \alpha_f \le \theta_{\widehat{f}}$. By $\theta(m)$ we mean $\theta_{\widehat{\varphi}}$, where φ is the scaling function corresponding to the mask m. The following result can be used for finding $\theta(m)$.

Proposition 2 *[8, Lemma 7.4.2 and Proposition 7.4.4] Suppose that some mask m is represented as $m(\omega) = \left(\cos\frac{\omega}{2}\right)^{L+1} m_c(\omega)$, where m_c is a pure mask. Then $\theta(m) = L + 1 + \theta(m_c)$ and $\theta(m_c) = \lim_{k\to\infty}\theta_k$, where*

$$\theta_k := -\frac{1}{k}\log_2 \|m_c(\omega)\cdots m_c(2^{k-1}\omega)\|_\infty. \tag{3}$$

3 Basic construction and conditions for a linear method of summation

Let us introduce a non-orthogonal mask of a new wavelet function. It is defined as the following 2π-periodic trigonometric polynomial

$$m_l(\omega) := \left(\cos\frac{\omega}{2}\right)^{2l} \frac{u_{n(l)}(m_l^M, \omega)}{u_{n(l)}(m_l^M, 0)}, \tag{4}$$

where

$$m_l^M(\omega) := \frac{m^M(\omega)}{\left(\cos\frac{\omega}{2}\right)^{2l}},\ l \in \mathbb{N},$$

m^M is a fixed Meyer mask, and trigonometric polynomial $u_{n(l)}(m_l^M, \cdot)$ is defined by a fixed linear method of summation for the function m_l^M.

Since m_l is a trigonometric polynomial and $m_l(0) = 1$, the infinite product $\prod_{j=1}^{\infty} m_l\left(\frac{\omega}{2^j}\right)$ converges absolutely and uniformly on an arbitrary compact set.

(If an infinite product is equal to zero we assume that it converges.) Thus the function m_l is a mask for a stable, but not orthogonal scaling function φ_l, the Fourier transform of φ_l is determined by the equality

$$\widehat{\varphi_l(\omega)} = \prod_{j=1}^{\infty} m_l\left(\frac{\omega}{2^j}\right). \tag{5}$$

The functions $\varphi_l(\cdot + k)$ for $k \in \mathbb{Z}$ form the Riesz basis in the closure of their linear span; this claim is straight corollary of Lemma 6 and Proposition 1. From estimation (16) it follows that the orthogonalizing factor

$$\Phi_l(\omega) := \sum_{k\in\mathbb{Z}} |\widehat{\varphi}_l(\omega + 2\pi k)|^2 \tag{6}$$

is well defined. Using the function Φ_l we define the Fourier transform of the orthogonal scaling function

$$\widehat{\varphi_l^{\perp}}(\omega) := \widehat{\varphi}_l(\omega)\Phi_l^{-0,5}(\omega), \tag{7}$$

the orthogonal mask

$$m_l^{\perp}(\omega) := m_l(\omega)\Phi^{0,5}(\omega)\Phi^{-0,5}(2\omega), \tag{8}$$

and, finally, the Fourier transform of the wavelet function

$$\widehat{\psi_l^{\perp}}(\omega) := e^{\frac{-i\omega}{2}}\overline{m_l^{\perp}}\left(\frac{\omega}{2}+\pi\right)\widehat{\varphi_l^{\perp}}\left(\frac{\omega}{2}\right). \tag{9}$$

Definition 1 *By* ***a quasispline wavelet function*** *we mean the function $\psi_l^{\perp}$, where the Fourier transform $\widehat{\psi_l^{\perp}}$ is defined by (9) and a non-orthogonal mask is defined by (4). The functions $\varphi_l^{\perp}$, $m_l^{\perp}$, φ_l, m_l defined by (7),(8),(5), and (4) respectively are called* ***a quasispline scaling function****, **a quasispline mask**, **a non-orthogonal quasispline scaling function**, and **a non-orthogonal quasispline mask** respectively.*

So for any fixed Meyer mask and for any fixed linear method of summation we get the sequence $(\psi_l)_{l\in\mathbb{N}}$ of quasispline wavelet functions, and the symbol l is a smoothness parameter (see Theorem 4).

In the remaining part of the article the following main Theorem will be proven.

Theorem 1 *Suppose that $\psi_l^\perp$ ($\varphi_l^\perp$) is a quasispline wavelet (scaling) function. Then*

1. *the functions $\varphi_l^\perp$ and $\psi_l^\perp$ decay exponentially at infinity (Theorem 5);*
2. *the functions $\widehat{\varphi_l^\perp}$ and $\widehat{\psi_l^\perp}$ decay as $O(\omega^{-l})$ at infinity, namely the Hölder exponents $\alpha_{\varphi_l^\perp}$ and $\alpha_{\psi_l^\perp}$ of the functions satisfy the inequalities*

   $$2l-1+\log_2\left(\frac{c}{1+\varepsilon(l)}\right) \le \alpha_{\varphi_l^\perp} \le 2l, \quad 2l-1+\log_2\left(\frac{c}{1+\varepsilon(l)}\right) \le \alpha_{\psi_l^\perp} \le 2l$$

   (Theorem 4);
3. *the uncertainty constants $\Delta^2_{\varphi_l^\perp}\Delta^2_{\widehat{\varphi_l^\perp}}$ ($\Delta^2_{\psi_l^\perp}\Delta^2_{\widehat{\psi_l^\perp}}$) of the quasispline scaling (wavelet) functions $\widehat{\varphi_l^\perp}$ ($\psi_l^\perp$) tend to those of the Meyer scaling (wavelet) function, namely*

   $$|\Delta^2_{\widehat{\varphi_l^\perp}} - \Delta^2_{\widehat{\varphi^M}}| = O\left(\max\{\mu(l),\, (4e^{2\omega_0})^{-2l+4\log_2\frac{1+\varepsilon(l)}{c}}\}\right),$$

   $$|\Delta^2_{\varphi_l^\perp} - \Delta^2_{\varphi^M}| = O\left(\max\{\mu(l),\, lC_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\}\right),$$

   $$|\Delta^2_{\widehat{\psi_l^\perp}} - \Delta^2_{\widehat{\psi^M}}| = O\left(\max\{\mu(l),\, lC_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\}\right),$$

   $$|\Delta^2_{\psi_l^\perp} - \Delta^2_{\psi^M}| = O\left(\max\{\mu(l),\, (4e^{2\omega_0})^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\}\right)$$

   as $l \to \infty$, where $C_0 := \frac{32\pi^2 e^{2\omega_0}}{27}$, others parameters are defined by (13) (Theorems 3, 6 and 7).

For the quasispline wavelet function $\psi_l^\perp$ to satisfy Theorem 1 and therefore to solve problem (1) it is sufficient to have three following conditions for the polynomials $u_{n(l)}(m_l^M, \cdot)$.

Theorem 2 *Suppose that there exists a sequence $n(l)$ for $l \in \mathbb{N}$ such that*

$$\|u_{n(l)}(m_l^M, \cdot) - m_l^M\|_C =: \alpha(l) = o(l^{-1}) \text{ as } l \to \infty, \tag{10}$$

$$\|u_{n(l)}((m_l^M)', \cdot) - (m_l^M)'\|_C =: \gamma(l) = o(1) \text{ as } l \to \infty, \tag{11}$$

$$u_{n(l)}(m_l^M, \pi) \ne 0, \tag{12}$$

then the corresponding quasispline scaling (7) and wavelet (9) functions satisfy the conditions of Theorem 1

De la Vallee Poussin means satisfy these conditions (for the proof see [6] p. 460, p.465, and p. 461 respectively).

By definition, put $u_l := u_{n(l)}(m_l^M, \cdot)$ and $u_{1,l} := u_{n(l)}((m_l^M)', \cdot)$, $u_{0,l} := u_l/u_l(0)$.

4 Convergence of frequency radii for the scaling functions

Lemma 1 $\|m_l - m^M\|_C \le K\alpha(l) = o(l^{-1})$ *as* $l \to \infty$, *where* $K := \frac{\|u_l\|_C}{\inf_{k\ge l}|u_k(0)|}$ *is bounded.*

Proof. Combining (4) and (10) we get

$$\|m_l - m^M\|_C = \left\|(\cos(\omega/2))^{2l}\frac{u_l}{u_l(0)} - m^M\right\|_C \le \left\|\frac{u_l}{u_l(0)} - m_l^M\right\|_C \le$$

$$\le \left\|\frac{u_l}{u_l(0)} - u_l\right\|_C + \|u_l - m_l^M\|_C \le \frac{\|u_l\|_C}{\inf_{k\ge l}|u_k(0)|}\|u_l(0)-1\|_C + \alpha(l) \le$$

$$\le \left(\frac{\|u_l\|_C}{\inf_{k\ge l}|u_k(0)|} + 1\right)\alpha(l).$$

Since $u_l(0) \to m_l^M(0) = 1$ as $l \to \infty$, it follows that for some $l_0 \in \mathbb{N}$ $\inf_{k\ge l_0}|u_k(0)| \ge c > 0$, therefore $\frac{\|u_l\|_C}{\inf_{k\ge l}|u_k(0)|}$ is bounded. $\Box$

From here we suppose that $l \ge l_0$. To simplify reading let us collect together notation of parameters using in estimations. So we get

$$\begin{array}{l}\mu(l) := l\alpha(l) + \gamma(l), \quad c := \inf_{l\ge l_0}|u_l(0)|, \quad \varepsilon(l) := \frac{\alpha(l)}{\|m_l^M\|_C}, \\ \frac{\pi}{3} \le \omega_0 < \frac{\pi}{2} \text{ (the parameter of the Meyer mask)}\end{array} \tag{13}$$

Lemma 2 $\|m_l' - (m^M)'\|_C = O(\mu(l))$ *as* $l \to \infty$.

Proof. Using Lemma 1, (2), and (11) we get

$$\left|\left(\left(\cos\frac{\omega}{2}\right)^{2l} u_l(\omega)\right)' - (m^M)'(\omega)\right| =$$

$$= \left|-l\left(\cos\frac{\omega}{2}\right)^{2l-1}\sin\frac{\omega}{2}\,u_l(\omega) + \left(\cos\frac{\omega}{2}\right)^{2l}u_l'(\omega) - (m^M)'(\omega)\right| =$$

$$= \left| -l \left(\cos \frac{\omega}{2} \right)^{2l-1} \sin \frac{\omega}{2} \left(m_l^M(\omega) + u_l(\omega) - m_l^M(\omega) \right) + \right.$$

$$\left. + \left(\cos \frac{\omega}{2} \right)^{2l} \left((m_l^M)'(\omega) + u_{1,l}(\omega) - (m_l^M)'(\omega) \right) - (m^M)'(\omega) \right| =$$

$$= \left| -l \tan \frac{\omega}{2} \, m^M(\omega) - l \left(\cos \frac{\omega}{2} \right)^{2l} \sin \frac{\omega}{2} \left(u_l(\omega) - m_l^M(\omega) \right) + \right.$$

$$+ \left(\cos \frac{\omega}{2} \right)^{2l} \cdot \frac{(m^M)'(\omega) \left(\cos \frac{\omega}{2} \right)^{2l} + l \left(\cos \frac{\omega}{2} \right)^{2l-1} \sin \frac{\omega}{2} \, m^M(\omega)}{\left(\cos \frac{\omega}{2} \right)^{4l}} +$$

$$\left. + \left(\cos \frac{\omega}{2} \right)^{2l} \left(u_{1,l}(\omega) - (m_l^M)'(\omega) \right) - (m^M)'(\omega) \right| =$$

$$\left| -l \left(\cos \frac{\omega}{2} \right)^{2l-1} \sin \frac{\omega}{2} \left(u_l(\omega) - m_l^M(\omega) \right) + \left(\cos \frac{\omega}{2} \right)^{2l} \left(u_{1,l}(\omega) - (m_l^M)'(\omega) \right) \right| =$$

$$= O(l\alpha(l) + \gamma(l)).$$

For m_l, we have

$$\left| m_l'(\omega) - (m^M)'(\omega) \right| = \left| \frac{\left(\left(\cos \frac{\omega}{2} \right)^{2l} u_l(\omega) \right)'}{u_l(0)} - (m^M)'(\omega) \right| \leq$$

$$\leq \left| \left(\left(\cos \frac{\omega}{2} \right)^{2l} u_l(\omega) \right)' \right| \left| u_l^{-1}(0) - 1 \right| + \left| \left(\left(\cos \frac{\omega}{2} \right)^{2l} u_l(\omega) \right)' - (m^M)'(\omega) \right| =$$

$$= \left(\|(m^M)'(\omega)\|_C + O(l\alpha(l) + \gamma(l)) \right) \frac{O(\alpha(l))}{c} + O(l\alpha(l) + \gamma(l)) =$$

$$= O(l\alpha(l) + \gamma(l)).\square$$

Lemma 3 *$\|\widehat{\varphi}_l - \widehat{\varphi^M}\|_{C[a,\,b]} = O(\mu(l))$ as $l \to \infty$ for any $a < b$, $a, b \in \mathbb{R}$. Parameter $\mu(l)$ is defined by (13)*

Proof. One can rewrite the proof of the Lemma from [6, Lemma 1]. It is sufficient to change the notation v_l by u_l and so on and to use the conditions (10), (11) instead of the property of the de la Vallee Poussin mean (see the formulas (4)-(7), (11), (12) [6]) $\square$.

Lemma 4 *$\left\|\widehat{\varphi}_l - \widehat{\varphi^M}\right\|_{L^2(\mathbb{R})} = O\left(\max\{\mu(l),\, (4e^{2\omega_0})^{-l+2\log_2 \frac{1+\varepsilon(l)}{c}}\}\right)$ as $l \to \infty$. Parameters are defined by (13).*

Proof. We claim that there exists a function ξ such that $\xi \in L^2(\mathbb{R})$ and $|\widehat{\varphi}_l(\omega)| \le \xi(\omega)$. The construction of the majorant can be rewritten with a inessential changes of notation from [6, Lemma 2]. So write the results. Denote

$$\widehat{\varphi_{0,l}}(\omega) := \prod_{j=1}^{\infty} \frac{u_l\left(\omega 2^{-j}\right)}{u_l(0)}. \tag{14}$$

Then under the assumption $|\omega| \ge 1$ we have

$$|\widehat{\varphi_{0,l}}(\omega)| \le |\omega|^{-2\theta(u_{0,l})} e^{2\omega_0(l+O(\mu(l)))} \le |\omega|^{2\log_2 \frac{1+\varepsilon(l)}{c}} e^{2\omega_0(l+O(\mu(l)))}. \tag{15}$$

So $|\widehat{\varphi}_l(\omega)|$ are majorized by the functions

$$|\widehat{\varphi}_l(\omega)| \le \xi_l(\omega) := \begin{cases} \left|\widehat{\varphi^M}(\omega)\right| + O(\mu(l)), & |\omega| \le 4e^{2\omega_0}, \\ e^{O(\mu(l))} |\omega|^{-l+2\log_2 \frac{1+\varepsilon(l)}{c}}, & |\omega| > 4e^{2\omega_0}. \end{cases} \tag{16}$$

Thus the function ξ may be defined as

$$\xi(\omega) := \begin{cases} \nu_1, & |\omega| \le 4e^{2\omega_0}, \\ \nu_2 |\omega|^{-l_1+2\log_2 \frac{1+\varepsilon(l)}{c}}, & |\omega| > 4e^{2\omega_0}, \end{cases}$$

where ν_1 and ν_2 are constants, $\nu_1, \nu_2 > 0$, $l_1 := \max\{l_0,\ 2\log_2 \frac{1+\varepsilon(l)}{c} + 2\}$. Then the convergence follows from the Lebesgue's dominated convergence Theorem and Lemma 3.

Let us estimate the rate of the convergence. If $|\omega| \ge 4e^{2\omega_0}$, then $\widehat{\varphi^M}(\omega) = 0$, so

$$\left\|\widehat{\varphi}_l - \widehat{\varphi^M}\right\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left|\widehat{\varphi}_l(\omega) - \widehat{\varphi^M}(\omega)\right|^2 d\omega = \int_{|\omega|<4e^{2\omega_0}} + \int_{|\omega|\ge 4e^{2\omega_0}} \le$$

$$\le 8e^{2\omega_0} \|\widehat{\varphi}_l - \widehat{\varphi^M}\|^2_{C[-4e^{2\omega_0},\, 4e^{2\omega_0}]} + e^{O(\mu(l))} \int_{|\omega|\ge 4e^{2\omega_0}} |\omega|^{-2l+4\log_2 \frac{1+\varepsilon(l)}{c}} d\omega =$$

$$= 8e^{2\omega_0} \|\widehat{\varphi}_l - \widehat{\varphi^M}\|^2_{C[-4e^{2\omega_0},\, 4e^{2\omega_0}]} + \frac{e^{O(\mu(l))} (4e^{2\omega_0})^{-2l+4\log_2 \frac{1+\varepsilon(l)}{c}+1}}{2l - 4\log_2 \frac{1+\varepsilon(l)}{c} - 1}.$$

This completes the proof of Lemma 4 $\square$.

Remark 1 *If we combine Lemma 3 and Lemma 4, we get* $\|\widehat{\varphi}_l - \widehat{\varphi^M}\|_{C(\mathbb{R})} = O\left(\max\{\mu(l),\, (4e^{2\omega_0})^{-l+2\log_2 \frac{1+\varepsilon(l)}{c}}\}\right)$.

Lemma 5 $|\Phi_l(\omega) - 1| = O\left(\max\{\mu(l),\, (4e^{2\omega_0})^{-2l+4\log_2 \frac{1+\varepsilon(l)}{c}}\}\right)$ *as* $l \to \infty$. *Parameters are defined by (13).*

Proof. Suppose $\omega \in [-\pi,\, \pi]$. Since φ^M is an orthogonal scaling function, we see that $\sum_{k\in\mathbb{Z}} |\widehat{\varphi^M}(\omega + 2\pi\omega)|^2 = 1$. Taking into account (16), we define $k_0 := [2e^{2\omega_0}/\pi + 1/2]$. Hence

$$|\Phi_l(\omega) - 1| = \left|\sum_{k\in\mathbb{Z}} |\widehat{\varphi}_l(\omega + 2\pi k)|^2 - \sum_{k\in\mathbb{Z}} |\widehat{\varphi^M}(\omega + 2\pi k)|^2\right| \leq$$

$$\leq \sum_{k\in\mathbb{Z}} \left| (\widehat{\varphi}_l(\omega + 2\pi k))^2 - \left(\widehat{\varphi^M}(\omega + 2\pi k)\right)^2 \right| = \sum_{|k|\leq k_0} + \sum_{|k|>k_0}.$$

Using Lemma 3 we get

$$\sum_{|k|\leq k_0} \leq (2k_0+1)(\sup_{|\omega|\leq 4e^{2\omega_0}} |\widehat{\varphi}_l(\omega) - \widehat{\varphi^M}(\omega)| + 2\sup_{|\omega|\leq 4e^{2\omega_0}} |\widehat{\varphi^M}(\omega)|)\times$$

$$\times \sup_{|\omega|\leq 4e^{2\omega_0}} |\widehat{\varphi}_l(\omega) - \widehat{\varphi^M}(\omega)| \leq O(\mu(l)).$$

Since $\widehat{\varphi^M} = 0$ as $|\omega| \leq 4e^{2\omega_0}$, (16), and the definition of k_0, we obtain

$$\sum_{|k|>k_0} \leq \sum_{|k|>k_0} e^{O(\mu(l))} |\omega + 2\pi k|^{-2l+4\log_2 \frac{1+\varepsilon(l)}{c}} = O\left((4e^{2\omega_0})^{-2l+4\log_2 \frac{1+\varepsilon(l)}{c}}\right).$$

Therefore,

$$|\Phi_l(\omega) - 1| = O\left(\max\{\mu(l),\, (4e^{2\omega_0})^{-2l+4\log_2 \frac{1+\varepsilon(l)}{c}}\}\right) \square.$$

Now let us prove the convergence of the frequency radii for the scaling function.

Theorem 3 $|\Delta^2_{\widehat{\varphi_l^\perp}} - \Delta^2_{\widehat{\varphi^M}}| = O\left(\max\{\mu(l),\, (4e^{2\omega_0})^{-2l+4\log_2 \frac{1+\varepsilon(l)}{c}}\}\right)$ *as* $l \to \infty$. *Parameters are defined by (13).*

Proof. Since the functions $\widehat{\varphi_l^\perp}$ and $\widehat{\varphi^M}$ are even, then $\omega_{0\widehat{\varphi_l^\perp}} = \omega_{0\widehat{\varphi^M}} = 0$, where $\omega_{0\widehat{\varphi_l^\perp}}$, $\omega_{0\widehat{\varphi^M}}$ are the frequency centers.

Taking into account Lemmas 3, 5, and the estimation (16) we have

$$|\Delta^2_{\widehat{\varphi_l^\perp}} - \Delta^2_{\widehat{\varphi^M}}| = \left| \int_{\mathbb{R}} \omega^2 \left(\left(\widehat{\varphi_l^\perp}\right)^2(\omega) - \left(\widehat{\varphi^M}\right)^2(\omega) \right) d\omega \right| \leq$$

$$\leq \int_{\mathbb{R}} \omega^2 \left| \frac{(\widehat{\varphi}_l)^2(\omega)}{\Phi_l(\omega)} - \left(\widehat{\varphi^M}\right)^2(\omega) \right| d\omega \leq \int_{|\omega|<4e^{2\omega_0}} + \int_{|\omega|\geq 4e^{2\omega_0}} \leq$$

$$\leq 16e^{4\omega_0} \int_{|\omega|<4e^{2\omega_0}} \left((\widehat{\varphi}_l)^2(\omega) \left| \frac{1}{\Phi_l(\omega)} - 1 \right| + \left| (\widehat{\varphi}_l)^2(\omega) - \left(\widehat{\varphi^M}\right)^2(\omega) \right| \right) d\omega +$$

$$+ \int_{|\omega|\geq 4e^{2\omega_0}} \omega^2 (\widehat{\varphi}_l)^2(\omega) \frac{1}{\Phi_l(\omega)} d\omega \leq 16e^{4\omega_0} \left(\|\Phi_l - 1\|_C \int_{|\omega|<4e^{2\omega_0}} \frac{(\widehat{\varphi}_l)^2(\omega)}{\Phi_l(\omega)} d\omega + \right.$$

$$\left. + \|\widehat{\varphi}_l - \widehat{\varphi^M}\|_{C[-4e^{2\omega_0},\, 4e^{2\omega_0}]} \int_{|\omega|<4e^{2\omega_0}} |\widehat{\varphi}_l(\omega) + \widehat{\varphi^M}(\omega)| \, d\omega \right) +$$

$$+ \frac{2e^{O(\mu(l))}(4e^{2\omega_0})^{-2l+4\log_2 \frac{1+\varepsilon(l)}{c}+3}}{\inf_{\omega,\, l} \Phi_l(\omega) \left(2l - 4\log_2 \frac{1+\varepsilon(l)}{c} - 3\right)}.$$

From Lemmas 3 and 5 it follows that the integrals

$$\int_{|\omega|<4e^{2\omega_0}} \frac{(\widehat{\varphi}_l)^2(\omega)}{\Phi_l(\omega)} d\omega, \quad \int_{|\omega|<4e^{2\omega_0}} |\widehat{\varphi}_l(\omega) + \widehat{\varphi^M}(\omega)| \, d\omega$$

are bounded. Hence

$$|\Delta^2_{\widehat{\varphi_l^\perp}} - \Delta^2_{\widehat{\varphi^M}}| = O\left(\max\{\mu(l),\, (4e^{2\omega_0})^{-2l+4\log_2 \frac{1+\varepsilon(l)}{c}}\} \right) + O(\mu(l)) +$$

$$+ O((4e^{2\omega_0})^{-2l+4\log_2 \frac{1+\varepsilon(l)}{c}}) = O\left(\max\{\mu(l),\, (4e^{2\omega_0})^{-2l+4\log_2 \frac{1+\varepsilon(l)}{c}}\} \right) \square.$$

5 The growth of the smoothness and the exponential decaying.

Lemma 6 *The polynomial $u_{0,l}$ is a pure mask.*

Proof. Let us use Proposition 1. Recall that $u_{0,l} = u_l/u_l(0)$. By the condition (10) and the inequality $\pi/3 \le \omega_0 < \pi/2$, where ω_0 is a parameter of the Meyer mask, we have $\sup_{[-\pi/3,\,\pi/3]} |u_l - m_l^M| = \sup_{[-\pi/3,\,\pi/3]} |u_l - (\cos\cdot/2)^{-2l}| = O(\alpha(l))$ as $l \to \infty$. Hence $u_l(\omega) \ne 0$ on the interval $\omega \in [-\pi/3,\, \pi/3]$. Therefore the polynomial u_l has no pair of symmetric zeros. If $B := \{b_1, \dots, b_n\}$ is a cyclic set and $b_1 = re^{i\xi}$, then $r = 1$, $\xi = \frac{2\pi k}{2^n - 1}$. If we suppose that B is a nontrivial cycle of the mask u_l then the set $\pi + \frac{2\pi k}{2^n-1}$ has to be roots of u_l. But it does not hold true because of $u_l(\omega) \ne 0$ on the interval $\omega \in [-\pi/3,\, \pi/3]$. Finally, the condition $u_l(\pi) \ne 0$ is postulated in (12). Then u_l has no the trivial cycle $\Box$.

Using Lemma 6 one can apply Proposition 2 to estimate smoothness of the non-orthogonal quasispline scaling function φ_l.

Lemma 7 *The following inequality holds true* $2l - 1 + \log_2\left(\frac{c}{1+\varepsilon(l)}\right) \le \alpha_{\varphi_l} \le 2l$. *Parameters are defined by (13).*

Proof. If we recall (10) and $c = \inf_{l \ge l_0} u_l(0)$, we get

$$\sup_\omega |u_{0,l}(\omega)| \le \frac{\sup_\omega |u_l(\omega)|}{c} \le \frac{(1+\varepsilon(l)) \sup_\omega |m_l^M(\omega)|}{c} \le \sup_\omega |f_{0,l}(\omega)|$$

for $\varepsilon(l) := \alpha(l)/\|m_l^M\|_C \to 0$ as $l \to \infty$, where $f_{0,l}$ is even 2π-periodic function and $f_{0,l}(\omega) := (1+\varepsilon(l))(\cos\omega/2)^{-2l}/c$ for $0 \le \omega \le \omega_1$ and $f_{0,l}(\omega) := 0$ for $\omega_1 < \omega \le \pi$. So we get $\theta_k(u_{0,l}) \ge \theta_k(f_{0,l})$.

The definition of $f_{0,l}$ yields

$$\left\| f_{0,l}(\omega) \cdots f_{0,l}(2^{k-1}\omega) \right\|_\infty = f_{0,l}(\omega_1) \cdots f_{0,l}(2^{-k+1}\omega_1) =$$

$$= \left(\cos\frac{\omega_1}{2} \cdots \cos\frac{\omega_1}{2^k}\right)^{-2l} \left(\frac{1+\varepsilon(l)}{c}\right)^k.$$

Then using Proposition 2 we have

$$\theta_k(f_{0,l}) = -\frac{1}{k}\log_2\left(\frac{1+\varepsilon(l)}{c}\right)^k - 2l\log_2\left|\cos\frac{\omega_1}{2} \cdots \cos\frac{\omega_1}{2^k}\right|^{-\frac{1}{k}} \to \log_2\left(\frac{c}{1+\varepsilon(l)}\right)$$

as $k \to \infty$. Passing to the limit, we use the identity $\prod_{j=1}^\infty \cos\frac{\omega}{2^j} = \frac{\sin\omega}{\omega}$. Therefore $\theta(u_{0,l}) \ge \log_2\left(\frac{c}{1+\varepsilon(l)}\right)$. For $u_{0,l}$ the multiplicity of the trivial cycle is

equal to $2l$. Hence $2l-1+\log_2\left(\frac{c}{1+\varepsilon(l)}\right) \le \alpha_{\varphi_l}$. By definition of the norm $\|\cdot\|_\infty$ we have $\|u_{0,l}(\omega)\dots u_{0,l}(2^{k-1}\omega)\|_\infty \ge u_{0,l}(0)\dots u_{0,l}(2^{k-1}\cdot 0) = 1$. Therefore Proposition 2 yields $\theta_k(u_{0,l}) \le 0$, then $\theta(u_{0,l}) \le 0$, thus $\alpha_{\varphi_l} \le 2l$. Finally, we obtain $2l-1+\log_2\left(\frac{c}{1+\varepsilon(l)}\right) \le \alpha_{\varphi_l} \le 2l$.$\Box$

Lemma 5 allows to extend the estimation of the smoothness to the orthogonal scaling and wavelet functions.

Theorem 4 *The following inequalities hold true*

$$2l-1+\log_2\left(\frac{c}{1+\varepsilon(l)}\right) \le \alpha_{\varphi_l^\perp} \le 2l, \quad 2l-1+\log_2\left(\frac{c}{1+\varepsilon(l)}\right) \le \alpha_{\psi_l^\perp} \le 2l.$$

Parameters are defined by (13).

Proof. It is sufficient to prove $\theta_{\widehat{\varphi_l}} = \theta_{\widehat{\varphi_l^\perp}} = \theta_{\widehat{\psi_l^\perp}}$. Using Lemma 5 we get $0 < c_1 \le \Phi_l(\omega) \le c_2 < \infty$. Therefore $c_2^{-0.5}|\widehat{\varphi_l}| \le |\widehat{\varphi_l^\perp}| \le c_1^{-0.5}|\widehat{\varphi_l}|$. Thus taking into account the definition of $\theta_{\widehat{f}}$ we get $\theta_{\widehat{\varphi_l}} = \theta_{\widehat{\varphi_l^\perp}}$.

Then the application of (9) yields

$$|\widehat{\psi_l^\perp}(\omega)| = \left| m_l\left(\frac{\omega}{2}+\pi\right)\Phi_l^{0,5}\left(\frac{\omega}{2}+\pi\right)\Phi_l^{-0,5}(\omega+2\pi)\,\widehat{\varphi}_l\left(\frac{\omega}{2}\right)\Phi_l^{-0,5}\left(\frac{\omega}{2}\right)\right|.$$

There exists an arbitrary large ω (for example, $\omega \in [-2\omega_0+2\pi(2k-1), 2\omega_0+2\pi(2k-1)]$, $k \in \mathbb{Z}$) such that $1-\alpha(l) \le m_l(\omega/2+\pi) \le 1+\alpha(l)$. Therefore for given ω we have $(1-\alpha(l))c_1^{0.5}c_2^{-1}|\widehat{\varphi}_l(\omega/2)| \le |\widehat{\psi_l^\perp}(\omega)| \le (1+\alpha(l))c_2^{0.5}c_1^{-1}|\widehat{\varphi}_l(\omega/2)|$. Finally, again taking into account the definition of $\theta_{\widehat{f}}$, we get $\theta_{\widehat{\varphi_l}} = \theta_{\widehat{\psi_l^\perp}}$ $\Box$.

Lemma 5 also allows to deduce exponential decay of the orthogonal scaling function $\varphi_l^\perp$ and the wavelet fucntion $\psi_l^\perp$.

Theorem 5 *The functions $\varphi_l^\perp$ and $\psi_l^\perp$ decay exponentially at infinity.*

Proof. Since m_l is a trigonometric polynomial, φ_l is compactly supported. Consequently taking into account Lemma 5, we deduce that $\varphi_l^\perp$ decays exponentially at infinity. So $\varphi_l^\perp(t) = O(e^{-\beta_2|t|})$, $\beta_2 > 0$. Fix $l \in \mathbb{N}$. The application of equality (9) yields $\psi_l^\perp(t) = \sum_{k\in\mathbb{Z}}(-1)^k h_{-k+1}\varphi_l^\perp(2t-k)$, where h_k are the Fourier coefficients of the function $m_l^\perp$. As $m_l^\perp$ is a rational trigonometric function and the denominator does not equal to 0 as $\omega \in \mathbb{R}$, then $h_k = O(e^{-\beta_1|k|})$, $\beta_1 > 0$. Therefore, we have

$$|\psi_l^\perp(t)| = \left|\sum_{k\in\mathbb{Z}}(-1)^k h_{-k+1}\varphi_l^\perp(2t-k)\right| \le \sum_{k\in\mathbb{Z}}\left|h_{-k+1}\varphi_l^\perp(2t-k)\right| \le$$

$$\leq A \sum_{k\in\mathbb{Z}} e^{-\beta_2|2t-k|-\beta_1|-k+1|},$$

where A is a constant. The application of the property of module and geometric series yields

$$\sum_{k\in\mathbb{Z}} e^{-\beta_2|2t-k|-\beta_1|-k+1|} =$$

$$= \frac{e^{\pm\beta_2-2\beta_2|t|}}{1-e^{-\beta_1-\beta_2}} + \frac{e^{\pm\beta_2}}{e^{\beta_2-\beta_1}-1}\left(e^{\beta_2|-2t+[2t]|-\beta_1|[2t]|} - e^{-2\beta_2|t|}\right) + \frac{e^{\kappa+\beta_2|2t-[2t]|-\beta_1|[2t]|}}{1-e^{-\beta_1-\beta_2}},$$

where $\kappa = -\beta_2$ as $t \geq 0$ and $\kappa = -\beta_1$ as $t < 0$. Therefore $\psi_l^{\perp} = O\left(e^{-\max\{\beta_1,\beta_2\}|2t|}\right)$.
$\square$

6 Convergence of time radii for the scaling functions

Lemma 8 *For any $-\infty < a < b < \infty$ it holds true $\left\|\widehat{\varphi}_l{}' - \widehat{\varphi^M}{}'\right\|_{C[a,\,b]} = O\left(\mu(l)\right)$ as $l \to \infty$. Parameter $\mu(l)$ is defined by (13).*

Proof. Using the definition of $\widehat{\varphi}_l$ we get

$$\left|\widehat{\varphi}_l{}'(\omega) - \widehat{\varphi^M}{}'(\omega)\right| = \left|\left(\prod_{j=1}^{\infty} m_l\left(\frac{\omega}{2^j}\right)\right)' - \left(\prod_{j=1}^{\infty} m^M\left(\frac{\omega}{2^j}\right)\right)'\right| =$$

$$= \left|\sum_{j_0=1}^{\infty} 2^{-j_0}\left(m_l'\left(\frac{\omega}{2^{j_0}}\right)\prod_{j=1,j\neq j_0}^{\infty} m_l\left(\frac{\omega}{2^j}\right) - (m^M)'\left(\frac{\omega}{2^{j_0}}\right)\prod_{j=1,j\neq j_0}^{\infty} m^M\left(\frac{\omega}{2^j}\right)\right)\right| \leq$$

$$\leq \sum_{j_0=1}^{\infty} 2^{-j_0}\left(\left|m_l'\left(\frac{\omega}{2^{j_0}}\right) - (m^M)'\left(\frac{\omega}{2^{j_0}}\right)\right|\prod_{j=1,j\neq j_0}^{\infty}\left|m^M\left(\frac{\omega}{2^j}\right)\right| + \right.$$

$$\left. + \left|m_l'\left(\frac{\omega}{2^{j_0}}\right)\right|\left|\prod_{j=1,j\neq j_0}^{\infty} m_l\left(\frac{\omega}{2^j}\right) - \prod_{j=1,j\neq j_0}^{\infty} m^M\left(\frac{\omega}{2^j}\right)\right|\right).$$

From Lemma 2 it follows that $\left|m_l'\left(\frac{\omega}{2^{j_0}}\right) - (m^M)'\left(\frac{\omega}{2^{j_0}}\right)\right| = O(\mu(l))$ and $\left|m_l'\left(\frac{\omega}{2^{j_0}}\right)\right| = \overline{M} + O(\mu(l))$, where $\overline{M} := \left\|(m^M)'\right\|_C$. Since $|m^M| \leq 1$, we have $\left|\prod_{j=1,j\neq j_0}^{\infty} m^M\left(\frac{\omega}{2^j}\right)\right| \leq 1$.

Taking into account Lemma 1 and the definition of $\widehat{\varphi}_l$, we obtain

$$\left|\prod_{j=1, j\neq j_0}^{\infty} m_l\left(\frac{\omega}{2^j}\right) - \prod_{j=1, j\neq j_0}^{\infty} m^M\left(\frac{\omega}{2^j}\right)\right| \leq \left|\prod_{j=1}^{j_0-1} m_l\left(\frac{\omega}{2^j}\right) - \prod_{j=1}^{j_0-1} m^M\left(\frac{\omega}{2^j}\right)\right| \left|\widehat{\varphi}_l\left(\frac{\omega}{2^{j_0}}\right)\right| +$$

$$+\left|\prod_{j=1}^{j_0-1} m^M\left(\frac{\omega}{2^j}\right)\right| \left|\widehat{\varphi}_l\left(\frac{\omega}{2^{j_0}}\right) - \widehat{\varphi^M}\left(\frac{\omega}{2^{j_0}}\right)\right|$$

Using (10) and the property of the Meyer mask $m^M \leq 1$ we get

$$\left|\prod_{j=1}^{j_0-1} m_l\left(\frac{\omega}{2^j}\right) - \prod_{j=1}^{j_0-1} m^M\left(\frac{\omega}{2^j}\right)\right| \leq$$

$$\leq \left|m_l\left(\frac{\omega}{2}\right) - m^M\left(\frac{\omega}{2}\right)\right| \prod_{j=2}^{j_0-1} m^M\left(\frac{\omega}{2^j}\right) + \left|m_l\left(\frac{\omega}{2}\right)\right| \left|\prod_{j=2}^{j_0-1} m_l\left(\frac{\omega}{2^j}\right) - \prod_{j=2}^{j_0-1} m^M\left(\frac{\omega}{2^j}\right)\right| \leq$$

$$\leq \|m_l - m^M\|_C + \left(1 + \|m_l - m^M\|_C\right) \left|\prod_{j=2}^{j_0-1} m_l\left(\frac{\omega}{2^j}\right) - \prod_{j=2}^{j_0-1} m^M\left(\frac{\omega}{2^j}\right)\right|.$$

Reiterating the procedure $j_0 - 2$ times we obtain

$$\left|\prod_{j=1}^{j_0-1} m_l\left(\frac{\omega}{2^j}\right) - \prod_{j=1}^{j_0-1} m^M\left(\frac{\omega}{2^j}\right)\right| = (1 + O(\alpha(l)))^{j_0-1} - 1.$$

From Lemma 3 and the definition of the Meyer scaling function it follows that $\left|\widehat{\varphi}_l\left(\frac{\omega}{2^{j_0}}\right)\right| = 1 + O(\mu(l))$ and $\left|\widehat{\varphi}_l\left(\frac{\omega}{2^{j_0}}\right) - \widehat{\varphi^M}\left(\frac{\omega}{2^{j_0}}\right)\right| = O(\mu(l))$. Finally, we note that $|m^M| \leq 1$, therefore $\left|\prod_{j=1}^{j_0-1} m^M\left(\frac{\omega}{2^j}\right)\right| \leq 1$.

Combining all the estimations together we obtain

$$\left|\widehat{\varphi}_l{}'(\omega) - \widehat{\varphi^M}{}'(\omega)\right| \leq O(\mu(l)) \sum_{j_0=1}^{\infty} 2^{-j_0} + \left(\overline{M} + O(\mu(l))\right)$$

$$\left(O(\mu(l)) \sum_{j_0=1}^{\infty} 2^{-j_0} + (1 + O(\mu(l))) \sum_{j_0=1}^{\infty} 2^{-j_0} \left((1 + O(\alpha(l)))^{j_0-1} - 1\right)\right) =$$

$$= O(\mu(l)) + \frac{O(\alpha(l))}{1 - O(\alpha(l))} = O(\mu(l)).$$

The next to last equality follows from the identity

$$\sum_{j_0=1}^{\infty} 2^{-j_0} \left((1+O(\alpha(l)))^{j_0-1} - 1\right) =$$

$$= \sum_{j_0=1}^{\infty} \left(\frac{1}{2} \left(\frac{1+O(\alpha(l))}{2} \right)^{j_0-1} - \frac{1}{2^{j_0}} \right) = \frac{O(\alpha(l))}{1-O(\alpha(l))} \square.$$

Lemma 9 $\left\| \widehat{\varphi}_l{}' - \widehat{\varphi^M}{}' \right\|_{L_2(\mathbb{R})} = O\left(\max\{\mu(l),\, l^{0.5} C_0^{-l+2\log_2 \frac{1+\varepsilon(l)}{c}}\} \right)$ *as* $l \to \infty$, *where* $C_0 := \frac{32\pi^2 e^{2\omega_0}}{27}$. *Other parameters are defined by (13).*

Proof. We prove the Lemma in a similar manner as Lemma 4. Let us find a majorant $\xi_1 \in L_2(\mathbb{R})$ for the function $\widehat{\varphi}_l{}'$. From the definition of $\widehat{\varphi}_l$, (14), and the identity $\sum_{j_0=1}^{\infty} 2^{-j_0} \tan \frac{\omega}{2^{j_0+1}} = \frac{2}{\omega} - \cot \frac{\omega}{2}$ it follows that

$$(\widehat{\varphi}_l)'(\omega) = \left(\prod_{j=1}^{\infty} m_l\left(\frac{\omega}{2^j}\right) \right)' = \sum_{j_0=1}^{\infty} 2^{-j_0} m_l'\left(\frac{\omega}{2^{j_0}}\right) \prod_{j=1, j\neq j_0}^{\infty} m_l\left(\frac{\omega}{2^j}\right) =$$

$$= \sum_{j_0=1}^{\infty} 2^{-j_0} \left(l \left(\cos \frac{\omega}{2^{j_0+1}} \right)^{2l-1} \left(-\sin \frac{\omega}{2^{j_0+1}} \right) \frac{u_l\left(\frac{\omega}{2^{j_0}}\right)}{u_l(0)} \prod_{j=1, j\neq j_0}^{\infty} \left(\cos \frac{\omega}{2^{j+1}} \right)^{2l} \frac{u_l\left(\frac{\omega}{2^j}\right)}{u_l(0)} + \right.$$

$$\left. + \left(\cos \frac{\omega}{2^{j_0+1}} \right)^{2l} \frac{u_{1,l}\left(\frac{\omega}{2^{j_0}}\right)}{u_l(0)} \prod_{j=1, j\neq j_0}^{\infty} \left(\cos \frac{\omega}{2^{j+1}} \right)^{2l} \frac{u_l\left(\frac{\omega}{2^j}\right)}{u_l(0)} \right) =$$

$$= \sum_{j_0=1}^{\infty} 2^{-j_0} l \left(-\tan \frac{\omega}{2^{j_0+1}} \right) \prod_{j=1}^{\infty} \left(\cos \frac{\omega}{2^{j+1}} \right)^{2l} \prod_{j=1}^{\infty} \frac{u_l\left(\frac{\omega}{2^j}\right)}{u_l(0)} +$$

$$+ \sum_{j_0=1}^{\infty} 2^{-j_0} \frac{u_{1,l}\left(\frac{\omega}{2^{j_0}}\right)}{u_l(0)} \prod_{j=1}^{\infty} \left(\cos \frac{\omega}{2^{j+1}} \right)^{2l} \prod_{j=1, j\neq j_0}^{\infty} \frac{u_l\left(\frac{\omega}{2^j}\right)}{u_l(0)} =$$

$$= l \left(\cot \frac{\omega}{2} - \frac{2}{\omega} \right) \left(\frac{\sin \omega/2}{\omega/2} \right)^{2l} \widehat{\varphi_{l,0}}(\omega) +$$

$$+ \left(\frac{\sin \omega/2}{\omega/2} \right)^{2l} \sum_{j_0=1}^{\infty} 2^{-j_0} \frac{u_{1,l}\left(\frac{\omega}{2^{j_0}}\right)}{u_l(0)} \prod_{j=1, j\neq j_0}^{\infty} \frac{u_l\left(\frac{\omega}{2^j}\right)}{u_l(0)} =: I_{1,l}(\omega) + \left(\frac{\sin \omega/2}{\omega/2} \right)^{2l} I_{2,l}(\omega).$$

If $|\omega| > 4e^{2\omega_0}$, then applying (15) and (16) for the first item we have

$$|I_{1,l}(\omega)| = \left| l\left(\cos\frac{\omega}{2} - \frac{2\sin\omega/2}{\omega}\right)\left(\frac{2}{\omega}\right)^{2l}(\sin\omega/2)^{2l-1}\widehat{\varphi_{l,0}}(\omega)\right| \le$$

$$\le Cle^{O(\mu(l))}|\omega|^{-l+2\log_2\frac{1+\varepsilon(l)}{2}}.$$

Let us estimate the second item.

$$I_{2,l}(\omega) = \sum_{j_0=1}^{\infty} 2^{-j_0}\frac{u_{1,l}\left(\frac{\omega}{2^{j_0}}\right)}{u_l(0)}\prod_{j=1}^{j_0-1}\frac{u_l\left(\frac{\omega}{2^j}\right)}{u_l(0)}\widehat{\varphi_{l,0}}\left(\frac{\omega}{2^{j_0}}\right).$$

Using (15) for $|\omega| \ge 4e^{2\omega_0}$ we get $\left|\widehat{\varphi_{l,0}}\left(\frac{\omega}{2^{j_0}}\right)\right| \le |\omega 2^{-j_0}|^{-2\theta(u_{0,l})}e^{2\omega_0(l+O(\mu(l)))}$. Using the condition (11), the definition of the function m_l^M, and the inequality $\pi/2 < \omega_1 \le 2\pi/3$ we get

$$\left|u_{1,l}\left(\frac{\omega}{2^{j_0}}\right)\right| \le \left|(m_l^M)'\left(\frac{\omega}{2^{j_0}}\right)\right| + O(\gamma(l)) \le$$

$$\le (m^M)'\left(\frac{\omega}{2^{j_0}}\right)\left(\cos\frac{\omega_1}{2^{j_0+1}}\right)^{-2l} + lm^M\left(\frac{\omega}{2^{j_0+1}}\right)\sin\frac{\omega_1}{2^{j_0+1}}\left(\cos\frac{\omega_1}{2^{j_0+1}}\right)^{-2l-1} +$$

$$+O(\gamma(l)) \le \left(\cos\frac{\omega_1}{2^{j_0+1}}\right)^{-2l}\left(\overline{M} + l\tan\frac{\omega_1}{2^{j_0+1}}\right) + O(\gamma(l)) \le$$

$$\le (4/3)^l\left(\overline{M} + l\sqrt{3}\right) + O(\gamma(l)).$$

Then taking into account condition (10), the properties of the Meyer mask $|m^M| \le 1$, $m^M(\omega) = 0$ as $\omega_1 \le |\omega| \le \pi$, and the inequality $\pi/2 < \omega_1 \le 2\pi/3$ we have

$$\left|\prod_{j=1}^{j_0-1}\frac{u_l\left(\frac{\omega}{2^j}\right)}{u_l(0)}\right| \le \prod_{j=1}^{j_0-1}\left(\frac{m^M(\omega 2^{-j})}{(\cos\omega 2^{-j-1})^{2l}} + \alpha(l)\right) \le \prod_{j=1}^{j_0-1}\left(\frac{1}{(\cos\omega_1 2^{-j-1})^{2l}} + \alpha(l)\right) \le$$

$$\le \prod_{j=1}^{j_0-1}\frac{a}{(\cos\omega_1 2^{-j-1})^{2l}} = a^{j_0-1}\prod_{j=1}^{\infty}(\cos\omega_1 2^{-j-1})^{-2l} =$$

$$= a^{j_0-1}\left(\frac{\omega_1/2}{\sin\omega_1/2}\right)^{2l} \le a^{j_0-1}\left(\frac{\omega_1}{\sqrt{2}}\right)^{2l} \le a^{j_0-1}\left(\frac{2\pi}{3\sqrt{2}}\right)^{2l},$$

where a is a majorant of the expression $1+\alpha(l)\left(\cos\omega_1 2^{-j-1}\right)^{2l}$, so it can be chosen $a<1.5$.

Collecting the estimations we obtain for $I_{2,l}(\omega)$

$$I_{2,l}(\omega) \le$$

$$\le \sum_{j_0=1}^{\infty} 2^{-j_0} \frac{\left(\frac{4}{3}\right)^l \left(\overline{M}+l\sqrt{3}\right)+O(\gamma(l))}{1-\alpha(l)} a^{j_0-1} \left(\frac{2\pi}{3\sqrt{2}}\right)^{2l} \left|\frac{\omega}{2^{j_0}}\right|^{-2\theta(u_{0,l})} e^{2\omega_0(l+O(\mu(l)))}.$$

Since $\log_2\left(\frac{c}{1+\varepsilon(l)}\right) \le \theta(u_{0,l}) \le 0$ and $a<1.5$, we get $|\omega|^{-2\theta(u_{0,l})} \le |\omega|^{2\log_2\frac{1+\varepsilon(l)}{c}}$ as $|\omega|\ge 1$, $2^{j_0\theta(u_{0,l})}\le 1$, and $\sum_{j_0=1}^{\infty} 2^{-j_0}a^{j_0-1}=(2-a)^{-1}$. So

$$I_{2,l}(\omega) \le \frac{e^{O(\mu(l))}\left(\overline{M}+l\sqrt{3}+(3/4)^l O(\gamma(l))\right)}{(1-\alpha(l))(2-a)} \left(\frac{8\pi^2 e^{2\omega_0}}{27}\right)^l |\omega|^{2\log_2\frac{1+\varepsilon(l)}{c}}$$

Thus we have for $|\omega| > \frac{32\pi^2 e^{2\omega_0}}{27}$

$$\left(\frac{\sin\omega/2}{\omega/2}\right)^{2l} I_{2,l}(\omega) \le (\sin\omega/2)^{2l}\, \frac{e^{O(\mu(l))}\left(\overline{M}+l\sqrt{3}+(3/4)^l O(\gamma(l))\right)}{(1-\alpha(l))(2-a)} \times$$

$$\times \left(\frac{32\pi^2 e^{2\omega_0}}{27}\right)^l |\omega|^{-2l+2\log_2\frac{1+\varepsilon(l)}{c}} \le C(l,\,\omega_0) l |\omega|^{-l+2\log_2\frac{1+\varepsilon(l)}{c}},$$

where $C(l,\,\omega_0) := e^{O(\mu(l))}\left(\overline{M}/l+\sqrt{3}+l^{-1}(3/4)^l O(\gamma(l))\right)(1-\alpha(l))^{-1}(2-a)^{-1}$ is bounded with respect to the parameters l and ω_0. Put $C(l,\,\omega_0)\le A$, A is a constant.

So if $|\omega| > C_0 := \frac{32\pi^2 e^{2\omega_0}}{27}$, we can estimate $(\widehat{\varphi}_l)'$ as follows $\left|(\widehat{\varphi}_l)'(\omega)\right| \le A\,l\,|\omega|^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}$.

Finally, using Lemma 8 one can define the functions $\xi_{1,l}$ such that

$$\left|(\widehat{\varphi}_l)'(\omega)\right| \le \xi_{1,l}(\omega) := \begin{cases} \left(\widehat{\varphi^M}\right)'(\omega)+O(\mu(l)), & |\omega| \le \frac{32\pi^2 e^{2\omega_0}}{27}, \\ A\,l\,|\omega|^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}, & |\omega| \ge \frac{32\pi^2 e^{2\omega_0}}{27}. \end{cases} \tag{17}$$

So the majorant ξ_1 is defined in the following way

$$\xi_1(\omega) := \begin{cases} \nu_1', & |\omega| \le \frac{32\pi^2 e^{2\omega_0}}{27}, \\ \nu_2'\,l\,|\omega|^{-l_1+2\log_2\frac{1+\varepsilon(l)}{c}}, & |\omega| \ge \frac{32\pi^2 e^{2\omega_0}}{27}. \end{cases}$$

where ν_1' and ν_2' are constants, $\nu_1', \nu_2' > 0$, $l_1 = \max\{l_0,\ 2\log_2 \frac{1+\varepsilon(l)}{c} + 2\}$ is defined in the proof of Lemma 4. Then the convergence follows from the Lebesgue's dominated convergence Theorem and Lemma 8.

Let us estimate the rate of the convergence. If $|\omega| \geq C_0$, then $\widehat{\varphi^M}(\omega) = 0$, so

$$\left\|\widehat{\varphi}_l{}' - \widehat{\varphi^M}{}'\right\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left|\widehat{\varphi}_l{}'(\omega) - \widehat{\varphi^M}{}'(\omega)\right|^2 d\omega = \int_{|\omega|<C_0} + \int_{|\omega|\geq C_0} \leq$$

$$\leq 2C_0 \|\widehat{\varphi}_l{}' - \widehat{\varphi^M}{}'\|^2_{C[-C_0,\,C_0]} + A^2 l^2 \int_{|\omega|\geq C_0} |\omega|^{-2l+4\log_2 \frac{1+\varepsilon(l)}{c}} d\omega =$$

$$= 2C_0 \|\widehat{\varphi}_l{}' - \widehat{\varphi^M}{}'\|^2_{C[-C_0,\,C_0]} + \frac{A^2 l^2 (C_0)^{-2l+4\log_2 \frac{1+\varepsilon(l)}{c}+1}}{2l - 4\log_2 \frac{1+\varepsilon(l)}{c} - 1} =$$

$$= O\left(\max\{\mu^2(l),\ l C_0^{-2l+4\log_2 \frac{1+\varepsilon(l)}{c}}\}\right).$$

This completes the proof of Lemma 9 $\square$.

Remark 2 *Using Lemma 8 and Lemma 9 we get*

$$\left\|\widehat{\varphi}_l{}' - \widehat{\varphi^M}{}'\right\|_{C(\mathbb{R})} = O\left(\max\{\mu(l),\ l^{0.5} C_0^{-l+2\log_2 \frac{1+\varepsilon(l)}{c}}\}\right).$$

Lemma 10 $\|\Phi_l'\|_C = O\left(\max\{\mu(l),\ l^{0.5}(4C_0 e^{2\omega_0})^{-l+2\log_2 \frac{1+\varepsilon(l)}{c}}\}\right)$ *as* $l \to \infty$. *Parameters are defined by (13).*

Proof. Taking into account the Definition Φ_l and the estimation (16) one can termwise differentiate the series, so

$$\Phi_l'(\omega) = \left(\sum_{k\in\mathbb{Z}} |\widehat{\varphi}_l(\omega + 2\pi k)|^2\right)' = \sum_{k\in\mathbb{Z}} \left(|\widehat{\varphi}_l(\omega + 2\pi k)|^2\right)'.$$

Since the Meyer scaling function is compactly supported and satisfies the property $\sum_{k\in\mathbb{Z}} \left|\widehat{\varphi^M}(\omega + 2\pi k)\right|^2 \equiv 1$, we get

$$\sum_{k\in\mathbb{Z}} \left(\left|\widehat{\varphi^M}(\omega + 2\pi k)\right|^2\right)' = \left(\sum_{k\in\mathbb{Z}} \left|\widehat{\varphi^M}(\omega + 2\pi k)\right|^2\right)' = (1)' = 0.$$

Suppose $|\omega| \le \pi$, then we obtain

$$|\Phi_l'(\omega)| \le 2\sum_{k\in\mathbb{Z}} \left|\widehat{\varphi}_l(\omega+2\pi k)\widehat{\varphi}_l{}'(\omega+2\pi k) - \widehat{\varphi^M}(\omega+2\pi k)\widehat{\varphi^M}{}'(\omega+2\pi k)\right| \le$$

$$\le 2\sum_{k\in\mathbb{Z}} |\widehat{\varphi}_l(\omega+2\pi k)| \left|\widehat{\varphi}_l{}'(\omega+2\pi k) - \widehat{\varphi^M}{}'(\omega+2\pi k)\right| +$$

$$+2\sum_{k\in\mathbb{Z}} \left|\widehat{\varphi^M}{}'(\omega+2\pi k)\right| \left|\widehat{\varphi}_l(\omega+2\pi k) - \widehat{\varphi^M}(\omega+2\pi k)\right| =: 2I_{3,l}(\omega) + 2I_{4,l}(\omega).$$

Using the parameter $k_0 = [2e^{2\omega_0}/\pi + 1/2]$ defined in the proof of Lemma 5 we get

$$I_{3,l}(\omega) = \sum_{|k|\le k_0} + \sum_{|k|>k_0}.$$

Taking into account Lemma 8, for the first sum we have

$$\sum_{|k|\le k_0} \le \|\widehat{\varphi}_l{}' - \widehat{\varphi^M}{}'\|_{C[-e^{2\omega_0},\, e^{2\omega_0}]} \sum_{|k|\le k_0} |\widehat{\varphi}_l(\omega+2\pi k)| = O(\mu(l)).$$

If we combine Remark 2 and the estimation (16), the second sum is

$$\sum_{|k|>k_0} \le O\left(\max\{\mu(l),\, l^{0.5} C_0^{-l+2\log_2 \frac{1+\varepsilon(l)}{c}}\}\right) \sum_{|k|>k_0} |\omega+2\pi k|^{-l+2\log_2 \frac{1+\varepsilon(l)}{c}} =$$

$$= O\left(\max\{\mu(l),\, l^{0.5} C_0^{-l+2\log_2 \frac{1+\varepsilon(l)}{c}}\}\right) (4e^{2\omega_0})^{-l+2\log_2 \frac{1+\varepsilon(l)}{c}}.$$

Estimate $I_{4,l}(\omega)$. Since $\operatorname{supp} \widehat{\varphi^M} = [-2\omega_1,\, -2\omega_0] \cup [2\omega_0,\, 2\omega_1]$, then $\widehat{\varphi^M}(\omega + 2\pi k) = 0$ as $k > 1$. So for the sum $I_{4,l}(\omega)$ we have

$$I_{4,l}(\omega) = \sum_{|k|\le 1} \left|\widehat{\varphi^M}{}'(\omega+2\pi k)\right| \left|\widehat{\varphi}_l(\omega+2\pi k) - \widehat{\varphi^M}(\omega+2\pi k)\right|.$$

Thus the application of Lemma 3 yields $I_{4,l}(\omega) = O(\mu(l))$. Finally, for Φ' we get

$$|\Phi'(\omega)| \le 2(I_{3,l}(\omega) + I_{4,l}(\omega)) = O\left(\max\{\mu(l),\, l^{0.5}(4C_0 e^{2\omega_0})^{-l+2\log_2 \frac{1+\varepsilon(l)}{c}}\}\right) \square.$$

Now let us prove the convergence of the time radii for the scaling function.

Theorem 6 $|\Delta^2_{\varphi_l^\perp} - \Delta^2_{\varphi^M}| = O\left(\max\{\mu(l),\, lC_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\}\right)$ *as* $l \to \infty$. *Parameters are defined by (13).*

Proof. If the function $\widehat{\varphi}$ is real-valued, then $\overline{\varphi}(t) = \varphi(-t)$. Hence the function $|\varphi|^2$ is even. So the time centre $t_{0\varphi} = 0$. Then the square of the time radius $\Delta^2_\varphi = \int_{\mathbb{R}} t^2|\varphi(t)|^2\,dt$. Using the property of the Fourier transform $\widehat{\varphi'}(\omega) = i\omega\widehat{\varphi}(\omega)$ we obtain $\Delta^2_\varphi = (2\pi)^{-1}\int_{\mathbb{R}} |(\widehat{\varphi})'(\omega)|^2\,d\omega$.

Since the functions $\widehat{\varphi_l^\perp}$, $\widehat{\varphi^M}$ are real-valued, then we have $t_{0\varphi_l^\perp} = t_{0\varphi^M} = 0$. So for the squares of the time radii we get

$$\Delta^2_{\varphi_l^\perp} = \frac{1}{2\pi}\int_{\mathbb{R}} |(\widehat{\varphi_l^\perp})'(\omega)|^2\,d\omega \text{ and } \Delta^2_{\varphi^M} = \frac{1}{2\pi}\int_{\mathbb{R}} |(\widehat{\varphi^M})'(\omega)|^2\,d\omega.$$

Then we have

$$\left|\Delta^2_{\varphi_l^\perp} - \Delta^2_{\varphi^M}\right| \le \frac{1}{2\pi}\int_{\mathbb{R}} \left| \left((\widehat{\varphi_l^\perp})'(\omega)\right)^2 - \left((\widehat{\varphi^M})'(\omega)\right)^2 \right| d\omega \le$$

$$\le \frac{1}{2\pi}\sup_{\omega\in\mathbb{R}}\left|(\widehat{\varphi_l^\perp})'(\omega) + (\widehat{\varphi^M})'(\omega)\right| \int_{\mathbb{R}} \left|(\widehat{\varphi_l^\perp})'(\omega) - (\widehat{\varphi^M})'(\omega)\right| d\omega$$

Applying Lemmas 5, 10 and Remarks 1, 2 we establish the boundedness of the supremum factor

$$\sup_{\omega\in\mathbb{R}}\left|\widehat{\varphi_l^\perp}'(\omega) + \widehat{\varphi^M}'(\omega)\right| \le \left\|\frac{\widehat{\varphi}_l{}'}{\Phi_l}\right\|_{C(\mathbb{R})} + \left\|\frac{\Phi_l'\widehat{\varphi}_l}{\Phi_l^2}\right\|_{C(\mathbb{R})} + \left\|\widehat{\varphi^M}'\right\|_{C(\mathbb{R})} \le$$

$$\frac{\left\|\widehat{\varphi^M}'\right\|_{C(\mathbb{R})} + \left\|\widehat{\varphi^M}' - \widehat{\varphi}_l{}'\right\|_{C(\mathbb{R})}}{1 - \|\Phi_l - 1\|_C} +$$

$$+\frac{\|\Phi_l'\|_C\left(\left\|\widehat{\varphi^M}\right\|_{C(\mathbb{R})} + \left\|\widehat{\varphi^M} - \widehat{\varphi}_l\right\|_{C(\mathbb{R})}\right)}{\left(1 - \|\Phi_l - 1\|_C\right)^2} + \left\|\widehat{\varphi^M}'\right\|_{C(\mathbb{R})} = O\left(\left\|\widehat{\varphi^M}'\right\|_{C(\mathbb{R})}\right).$$

Applying the same Lemmas 5, 10 and Remarks 1, 2 we get the convergence to 0 of the integral

$$\int_{\mathbb{R}}\left|\widehat{\varphi_l^\perp}'(\omega) - \widehat{\varphi^M}'(\omega)\right| d\omega \le \int_{\mathbb{R}}\left|\frac{\widehat{\varphi}_l{}'(\omega) - \widehat{\varphi^M}'(\omega)\Phi_l(\omega)}{\Phi_l(\omega)}\right| d\omega + \int_{\mathbb{R}}\left|\frac{\widehat{\varphi}_l(\omega)\Phi_l'(\omega)}{\Phi_l^2(\omega)}\right| d\omega \le$$

$$\leq \frac{1}{1-\|\Phi_l - 1\|_C}\int_{\mathbb{R}}\left|\widehat{\varphi_l}'(\omega)-\widehat{\varphi^M}'(\omega)\right|+\left|\widehat{\varphi^M}'(\omega)\right||1-\Phi_l(\omega)|\,d\omega+$$

$$+\frac{\|\Phi'\|_C}{(1-\|\Phi_l-1\|_C)^2}\int_{\mathbb{R}}|\widehat{\varphi_l}(\omega)|\,d\omega \leq$$

$$\leq \frac{1}{1-\|\Phi_l-1\|_C}\left(\int_{|\omega|\leq C_0}\left|\widehat{\varphi_l}'(\omega)-\widehat{\varphi^M}'(\omega)\right|\,d\omega+\int_{|\omega|>C_0}\left|\widehat{\varphi_l}'(\omega)-\widehat{\varphi^M}'(\omega)\right|\,d\omega\right)+$$

$$+\frac{\|1-\Phi\|_C}{1-\|\Phi_l-1\|_C}\int_{\mathbb{R}}\left|\widehat{\varphi^M}'(\omega)\right|\,d\omega+O\left(\|\Phi'\|_C\right)\leq$$

$$\leq \frac{\left\|\widehat{\varphi_l}'-\widehat{\varphi^M}'\right\|_{C[-C_0,\,C_0]}}{1-\|\Phi_l-1\|_C}+\frac{2AlC_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}+1}}{(1-\|\Phi_l-1\|_C)\left(l-2\log_2\frac{1+\varepsilon(l)}{c}-1\right)}+$$

$$+O\left(\|\Phi_l-1\|_C\right)+O\left(\|\Phi'\|_C\right)=O\left(\max\{\mu(l),\,lC_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\}\right)\square.$$

In Lemmas 8-10 and Theorem 6 we apply without proof the formula $\left(\prod_{j=1}^{\infty} m_l(\omega 2^{-j})\right)' = \sum_{j_0=1}^{\infty} 2^{-j_0} m_l'(\omega 2^{-j_0})\prod_{j=1, j\neq j_0}^{\infty} m_l(\omega 2^{-j})$. Establish it in the following

Lemma 11 *For any a, b such that $-\infty < a < b < \infty$ we have*

$$\left\|\left(\prod_{j=1}^{\infty} m_l(\omega 2^{-j})\right)'-\left(\prod_{j=1}^{n} m_l(\omega 2^{-j})\right)'\right\|_{C[a,\,b]}\longrightarrow 0$$

as $n\to\infty$, where $\left(\prod_{j=1}^{\infty} m_l(\omega 2^{-j})\right)'$ is the notation for the series $\sum_{j_0=1}^{\infty} 2^{-j_0} m_l'(\omega 2^{-j_0})\prod_{j=1, j\neq j_0}^{\infty} m_l(\omega 2^{-j})$.

Proof. Using the introduced notation we have

$$\left|\left(\prod_{j=1}^{\infty} m_l\left(\frac{\omega}{2^j}\right)\right)'-\left(\prod_{j=1}^{n} m_l\left(\frac{\omega}{2^j}\right)\right)'\right|=$$

$$=\left|\sum_{j_0=1}^{\infty}2^{-j_0}m_l'\left(\frac{\omega}{2^{j_0}}\right)\prod_{j=1,j\neq j_0}^{\infty}m_l\left(\frac{\omega}{2^j}\right)-\sum_{j_0=1}^{n}2^{-j_0}m_l'\left(\frac{\omega}{2^{j_0}}\right)\prod_{j=1,j\neq j_0}^{n}m_l\left(\frac{\omega}{2^j}\right)\right|\leq$$

$$\leq \left| \sum_{j_0=n+1}^{\infty} 2^{-j_0} m_l' \left(\frac{\omega}{2^{j_0}}\right) \prod_{j=1, j\neq j_0}^{\infty} m_l\left(\frac{\omega}{2^j}\right) \right| +$$

$$+ \left| \sum_{j_0=1}^{n} 2^{-j_0} m_l' \left(\frac{\omega}{2^{j_0}}\right) \left(\prod_{j=1, j\neq j_0}^{\infty} m_l\left(\frac{\omega}{2^j}\right) - \prod_{j=1, j\neq j_0}^{n} m_l\left(\frac{\omega}{2^j}\right) \right) \right| =: I_{5,n}(\omega) + I_{6,n}(\omega).$$

The application of the Lagrange Theorem and Lemma 2 yields $|m_l(\omega)| \leq 1 + A|\omega|$, where A is a constant. Hence

$$\left| \prod_{j=1}^{j_0-1} m_l\left(\frac{\omega}{2^j}\right) \right| \leq \prod_{j=1}^{j_0-1} \left(1 + A|\omega| 2^{-j}\right) = e^{\sum_{j=1}^{j_0-1} \ln(2^j + A|\omega|) - \ln 2^j} \leq e^{A|\omega| \sum_{j=1}^{j_0-1} \frac{1}{2^j}} \leq e^{A|\omega|}.$$

So using additionally Lemmas 2 and 3 for the first sum we get

$$I_{5,n}(\omega) \leq \sum_{j_0=n+1}^{\infty} 2^{-j_0} \left| m_l' \left(\frac{\omega}{2^j}\right) \prod_{j=1}^{j_0-1} m_l\left(\frac{\omega}{2^j}\right) \widehat{\varphi}_l\left(\frac{\omega}{2^{j_0}}\right) \right| \leq$$

$$\leq \left(\left\| (m^M)' \right\|_C + \left\| m_l' - (m^M)' \right\|_C \right) e^{A|\omega|} \left(\left\| \widehat{\varphi^M} \right\|_{C(\mathbb{R})} + \left\| \widehat{\varphi^M} - \widehat{\varphi}_l \right\|_{C(\mathbb{R})} \right) 2^{-n},$$

where all factors are bounded as $a \leq \omega \leq b$, $l \in \mathbb{N}$. Thus $I_{5,n}(\omega) \to 0$ as $n \to \infty$.

For the second sum $I_{6,n}(\omega)$ we obtain

$$I_{6,n}(\omega) = \left| \sum_{j_0=1}^{n} 2^{-j_0} m_l' \left(\frac{\omega}{2^{j_0}}\right) \prod_{j=1, j\neq j_0}^{n} m_l\left(\frac{\omega}{2^j}\right) \left(\widehat{\varphi}_l\left(\frac{\omega}{2^n}\right) - 1 \right) \right|.$$

Since the function $\widehat{\varphi}_l$ is continuous, $\widehat{\varphi}_l(0) = 1$, and $a \leq |\omega| \leq b$, it follows that $\left|\widehat{\varphi}_l\left(\frac{\omega}{2^n}\right) - 1\right| = \varepsilon_1(n)$, $\varepsilon_1(n) \to 0$ as $n \to \infty$. So we get

$$I_{6,n}(\omega) \leq \left(1 - \frac{1}{2^{n+1}}\right) \left(\left\| (m^M)' \right\|_C + \left\| m_l' - (m^M)' \right\|_C \right) e^{A|\omega|} \varepsilon_1(n),$$

where all factors are bounded as $a \leq \omega \leq b$, $l \in \mathbb{N}$. Thus $I_{5,n}(\omega) \to 0$ as $n \to \infty$ $\square$.

7 Convergence of time and frequency radii for the wavelet functions

Theorem 7 *$|\Delta^2_{\widehat{\psi_l^\perp}} - \Delta^2_{\widehat{\psi^M}}| = O\left(\max\{\mu(l),\, lC_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\}\right)$, $|\Delta^2_{\psi_l^\perp} - \Delta^2_{\psi^M}| = O\left(\max\{\mu(l),\, (4e^{2\omega_0})^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\}\right)$ as $l\to\infty$. Parameters are defined by (13).*

Proof. The equality (9) shows that $\widehat{\psi_l^\perp}$ is even. The function $\widehat{\psi^M}$ is also even. Therefore $\omega_{0\widehat{\psi_l^\perp}} = \omega_{0\widehat{\psi^M}} = 0$ and $t_{0\psi_l^\perp} = t_{0\psi^M} = 1/2$. The mask $m_l^\perp$ is real-valued function. Therefore, using the stucture of formula (9) and applying Lemmas 1, 5, and Theorem 3 we get for the frequency radii

$$|\Delta^2_{\widehat{\psi_l^\perp}} - \Delta^2_{\widehat{\psi^M}}| =$$

$$= \left|\int_{\mathbb{R}} \omega^2 \left((m_l^\perp)^2\left(\frac{\omega}{2}+\pi\right)\left(\widehat{\varphi_l^\perp}\right)^2\left(\frac{\omega}{2}\right) - (m^M)^2\left(\frac{\omega}{2}+\pi\right)\left(\widehat{\varphi^M}\right)^2\left(\frac{\omega}{2}\right)\right) d\omega\right| \le$$

$$\le \|(m_l^\perp)^2\|_C \int_{\mathbb{R}} \omega^2 \left|\left(\widehat{\varphi_l^\perp}\right)^2\left(\frac{\omega}{2}\right) - \left(\widehat{\varphi^M}\right)^2\left(\frac{\omega}{2}\right)\right| d\omega +$$

$$+\|(m_l^\perp)^2 - (m^M)^2\|_C \int_{\mathbb{R}} \omega^2 \left(\widehat{\varphi^M}\right)^2\left(\frac{\omega}{2}\right) d\omega =$$

$$= O\left(|\Delta^2_{\widehat{\varphi_l^\perp}} - \Delta^2_{\widehat{\varphi^M}}|\right) + O\left(\|\Phi_l - 1\|_C\right) + O\left(\|m_l - m^M\|_C\right) =$$

$$= O\left(\max\{\mu(l),\, lC_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\}\right).$$

Going on to the time radii we use the identity $\Delta_f^2 = \int_{\mathbb{R}} t^2|f(t)|^2\,dt - t_{0f}^2$. Then applying Lemmas 2, 10, 9 and Remark 2 we obtain

$$2\pi|\Delta^2_{\psi_l^\perp} - \Delta^2_{\psi^M}| = \left|\int_{\mathbb{R}} \left(\widehat{\psi_l^\perp}'\right)^2(\omega) - \left(\widehat{\psi^M}'\right)^2(\omega)\,d\omega\right| \le$$

$$\le \left\|\widehat{\psi_l^\perp}' + \widehat{\psi^M}'\right\|_{C(\mathbb{R})} \int_{\mathbb{R}} \left|\widehat{\psi_l^\perp}'(\omega) - \widehat{\psi^M}'(\omega)\right| d\omega \le$$

$$\le A \int_{\mathbb{R}} \left| {m_l^\perp}' \widehat{\varphi_l^\perp} + m_l^\perp \widehat{\varphi_l^\perp}' - i m_l^\perp \widehat{\varphi_l^\perp} - {m^M}' \widehat{\varphi^M} - m^M \widehat{\varphi^M}' + i m^M \widehat{\varphi^M} \right| \le$$

$$\leq A\left(\| {m_l^{\perp}}' - i m_l^{\perp}\|_C \int_{\mathbb{R}} \left|\widehat{\varphi_l^{\perp}} - \widehat{\varphi^M}\right| + \| {m_l^{\perp}}' - {m^M}' \|_C \int_{\mathbb{R}} \left|\widehat{\varphi^M}\right| + \right.$$

$$\left. + \|m_l^{\perp}\|_C \int_{\mathbb{R}} \left|\widehat{\varphi_l^{\perp}}' - \widehat{\varphi^M}'\right| + \|m_l^{\perp} - m^M\|_C \int_{\mathbb{R}} \left|\widehat{\varphi^M}' - i\widehat{\varphi^M}\right|\right) =: I_{7,l} + I_{8,l} + I_{9,l} + I_{10,l}.$$

The application of Lemmas 3, 4, and 5 yields

$$I_{7,l} = O\left(\|\Phi_l - 1\|_C\right) + O\left((4e^{2\omega_0})^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\right) = O\left(\max\{\mu(l),\, (4e^{2\omega_0})^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\}\right).$$

Using the Definition of $m_l^{\perp}$, Lemmas 10, 5, and 2 we get

$$I_{8,l} = O\left(\|\Phi_l'\|_C\right) + O\left(\|\Phi_l - 1\|_C\right) + O\left(\|m_l' - {m^M}'\|_C\right) =$$

$$= O\left(\max\{\mu(l),\, (4e^{2\omega_0})^{-2l+4\log_2\frac{1+\varepsilon(l)}{c}}\}\right).$$

From Theorem 6 it follows that

$$I_{9,l} = O\left(\max\{\mu(l),\, l C_0^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\}\right).$$

Finally, using Lemmas 5 and 1 we get

$$I_{10,l} = O\left(\|\Phi_l - 1\|_C\right) + O\left(\|m_l - m^M\|_C\right) = O\left(\max\{\mu(l),\, (4e^{2\omega_0})^{-2l+4\log_2\frac{1+\varepsilon(l)}{c}}\}\right).$$

Thus, collecting the estimations we obtain

$$|\Delta^2_{\psi_l^{\perp}} - \Delta^2_{\psi^M}| = O\left(\max\{\mu(l),\, (4e^{2\omega_0})^{-l+2\log_2\frac{1+\varepsilon(l)}{c}}\}\right) \square.$$

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